EQUIVARIANT CHERN CLASSES OF SINGULAR ALGEBRAIC VARIETIES WITH GROUP ACTIONS

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Abstract. We define the equivariant Chern-Schwartz-MacPherson class of a possibly singular algebraic G-variety over the base field $\mathbb{C}$, or more generally over a field of characteristic 0. In fact, we construct a natural transformation C_*^G from the G-equivariant constructible function functor $\mathcal{F}^G$ to the G-equivariant homology functor H_*^G or A_*^G (in the sense of Totaro-Edidin-Graham). This C_*^G may be regarded as MacPherson's transformation for (certain) quotient stacks. We discuss on other type Chern classes and applications. The Verdier-Riemann-Roch formula takes a key role throughout.

1. Introduction

For a possibly singular complex algebraic variety X there are several kinds of "Chern classes" of X available. These "Chern classes" of X live in appropriate *homology groups* of X, which satisfy "the normalization property" that if X is non-singular, then it coincides with the Poincaré dual to the ordinary Chern class of the tangent bundle TX.

The Chern-Schwartz-MacPherson class is one of them. R. MacPherson [19] constructed the class to solve the so-called Grothendieck-Deligne conjecture: Actually he proved the existence of a unique natural transformation $C_* : \mathcal{F}(X) \to H_{2*}(X;\mathbb{Z})$ from the abelian group $\mathcal{F}(X)$ of *constructible functions* over X to the homology group (of even dimension) so that if X is nonsingular, then $C_*(1\!\!1_X) = c(TX) \frown [X]$ where $1\!\!1_X$ is the characteristic function, $1\!\!1_X(x) = 1$ $(x \in X)$. Independently M. H. Schwartz [26] had introduced obstruction classes (defined in a local cohomology) for the extension of stratified radial vector frames over X, and it is shown ([4]) that both classes coincide, so $C_*(1\!\!1_X)$ is often denoted by $C^{SM}(X)$. In a purely algebraic context, MacPherson's transformation is also formulated as $C_* : \mathcal{F}(X) \to A_*(X)$, the value being in the Chow group of cycles modulo rational equivalence, for embeddable schemes (separated and of finite type) over arbitrary base field k of characteristic 0. That was done by G. Kennedy [16] using the groups of *Lagrangian cycles* (cf. [12], [24]), which is isomorphic to $\mathcal{F}(X)$ in a certain way. In the complex analytic context, MacPherson's theory

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is also verified: for instance, the crucial step in [19], *the graph construction*, is proved in [17] in the analytic setting. Besides, the Lagrangian cycle approach in the complex differential geometry [11] and Schwartz's approach within the Chern-Weil theory [3] have been also achieved.

In this paper we think of a G-version of the Chern-Schwartz-MacPherson class for algebraic G-varieties X. Our main aim is to focus the elementary (or formal) construction of the equivariant version of C_* as well G-versions of $\mathcal{F}(X)$ and $H_*(X)$ (or $A_*(X)$). So for the simplicity we discuss basically in the complex context like as the original [19]: Then we use the singular cohomology and the Borel-Moore homology, simply denoted by $H_*(X)$, of the underlying *analytic* space (denoted by the same letter X for short). However, after suitable changes, the reader can read them as in the algebraic context ([16]) with the use of (operational) Chow rings and Chow groups: Then a *scheme* is assumed to be separated and of finite type over k of characteristic 0, and a *variety* is an irreducible and reduced such scheme.

As known, for a topological group G, Borel's equivariant cohomology of a G-space X is defined by

$$H^*_G(X) = H^*(X \times_G EG),$$

where $EG \to BG$ is the universal principal bundle over the classifying space of G. As a counterpart in algebraic geometry, for reductive linear algebraic group G, the *G-equivariant homology group* $H^G_*(X)$ ($A^G_*(X)$) of a G-variety X is defined in Edidin-Graham [6] using the algebraic approximation of BG given by Totaro [28]. From the same viewpoint, we introduce the abelian group $\mathcal{F}^G(X)$ of *G-equivariant constructible functions* over X (that is, roughly, constructible functions over $X \times_G EG$ whose supports have finite codimension). In particular the group $\mathcal{F}^G_{inv}(X)$ of *G-invariant* constructible functions over X becomes a subgroup of $\mathcal{F}^G(X)$ by a natural identification. Both of $\mathcal{F}^G$ and H^G_* become *covariant* functors for the category of G-varieties and proper G-morphisms (see subsections 2.4 and 2.6).

From now on we assume that a G-variety (scheme) X has a closed equivariant embedding into some G-nonsingular varieties, and when we emphasize it, we say such X is *G-embeddable* for short. We show the following theorem for G-embeddable varieties:

Theorem 1.1. *Let G be a complex reductive linear algebraic group. For the category of complex algebraic G-varieties X and proper G-morphisms, there is a natural transformation of covariant functors*

$$C^G_* : \mathcal{F}^G(X) \to H^G_*(X)$$

such that if X is non-singular, then $C^G_*(1\!\!1_X) = c^G(TX) \frown [X]_G$ where $c^G(TX)$ is G-equivariant total Chern class of the tangent bundle of X. The natural transformation C^G_* is unique in a certain sense.

To be precise, we mean by *a natural transformation* that C^G_* satisfies that (i): $C^G_*(\alpha + \beta) = C^G_*(\alpha) + C^G_*(\beta)$ and (ii): $f^G_* C^G_* = C^G_* f^G_*$ for any proper G-

morphism $f : X \to Y$. The precise statement of the "uniqueness" of C_*^G is seen in the subsection 3.2 (b).

Remark 1.2. *Theorem* 1.1 *is also true over the base field* k *of characteristic* 0*, (at least) for quasi-projective schemes* X *with linearlized* G*-action; then we have a natural transformation* $C_*^G : \mathcal{F}^G(X) \to A_*^G(X)$ *which satisfies the normalization property (see the subsection* 2.2 *and the proof of Theorem* 1.1 *given in §3). This* C_*^G *is naturally regarded as the extension of MacPherson's Chern class theory to the category of quotient stacks,* $C_* : \mathcal{F}([X/G]) \to A_*([X/G])$ *(Theorem* 3.5*).*

Definition 1.3. *The* G*-equivariant Chern-Schwartz-MacPherson class of a* G*-variety* X *is defined by* $C_G^{SM}(X) := C_*^G(1\!\!1_X)$*.*

Besides of $1\!\!1_X$ and $C_G^{SM}(X)$, we can take other kinds of "canonical constructible functions" over X and the corresponding "canonical Chern-SM classes" of X. That will be discussed in §6.

The rest of this paper is organized as follows:

In §2 we will review some basic materials from [28] and [6] but in a slightly different form. Groups $\mathcal{F}^G(X)$ and $H_*^G(X)$ ($A_*^G(X)$) are defined to be the inductive limit of abelian groups via very simple "Radon transforms" (labeled by $(*_F)$ and $(*_H)$). In §3 our C_*^G is given as the limit of "MacPherson's transformation for topological Radon transforms" studied in [7]. Then Theorem 1.1 automatically follows. We remark that this construction is very related to the "proconstruction" of C_* for *provarieties* (projective limits of varieties) given by Yokura [31] (Remark 3.3). The last subsection 3.4 of §3 is devoted to the interpretation of C_*^G in terms of quotient stacks.

In §4 we note some useful properties of our C_*^G, for instance, the equivariant versions of *Verdier-Riemann-Roch formula* (written by VRR formula for short) for smooth morphisms ([10], [30], and also [25] for local complete intersection morphisms). That is a Riemann-Roch type theorem saying the compatibility of the transformation C_* with certain *pullbacks* (i.e., contravariant operation) of constructible functions and homologies. In fact, the simplest VRR formula is involved in our construction of C_*^G itself (the square (2) in the proof of Lemma 3.1).

The equivariant Chern-Mather class $C_G^M(X)$ is introduced in §5. As known, the Chern-Mather class $C^M(X)$ is a key factor in the construction of (ordinary) MacPherson's class, which is roughly the Chern class of limiting tangent spaces of the regular part of X. In fact $C^{SM}(X)$ is expressed by $C^M(X)$ plus a certain linear combination of $C^M(W)$'s of subvarieties W in the singular locus of X. We show the equivariant version of such relations. Then some simple properties of C_*^G become clearer: for instance, the restriction of C_*^G to a fibre of the universal principal bundle $X \times_G EG \to BG$ recovers the ordinary MacPherson transformation C_*.

In §6 and §7, we discuss on apparently a bit different two kinds of applications, which generalize *orbifold Euler characteristics* (cf. [13], [5]) and *Thom*

polynomials (cf. [27], [14], [8]), respectively. As to the former topic, *the canonical quotient Chern classes* are introduced, which reflect some commutator structure of the group action. In [22] we will apply this theory to typical examples such as symmetric products, and obtain generating functions of the quotient Chern classes whose constant terms provide well-known generating functions of (orbifold) Euler characteristics. As to the latter topic, a Thom polynomial is roughly saying the G-Poincaré dual to an invariant subvariety of a G-nonsingular variety. As a simple generalization, we study the G-Poincaré dual to the "Segre-version" of our equivariant Chern class. Our Theorem 7.5 is motivated by the formula of Parusiński-Pragacz [23] (Theorem 2.1) for degeneracy loci of generic vector bundle morphisms.

Consequently, it can be viewed that these two applications deal with a unified "Chern class version" of *Euler characteristics* and *fundamental classes* arising in some "G-classification theory".

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2. Classifying space and the Borel construction

In 2.1 – 2.5 we pick up some definitions and properties from [28] and [6] Note again that H_* can be replaced by the Chow group A_* in the algebraic context. In 2.6 the equivariant constructible function is defined.

2.1. Totaro's construction of BG. Let G be a complex reductive linear algebraic group of dimension g. Take an l-dimensional representation V of G with a G-invariant Zariski closed subset S in V so that G acts on $U := V - S$ freely. It is possible to take V and S so that the quotient $U \to U/G$ becomes an algebraic principal G-bundle over a quasi-projective variety, and that the codimension of S is sufficiently high. Actually this is achieved by a similar construction of Grassmanian varieties (Remark 1.4 of [28]). Let $I(G)$ be the collection of Zariski open sets $U = V - S$ where V is a representation and S is a closed subset of V with properties just as mentioned. We put a partial order on $I(G)$: we say $U(= V - S) < U'(= V' - S')$ if $\operatorname{codim}_V S < \operatorname{codim}_{V'} S'$ and there is an G-equivariant linear inclusion $V \to V'$ sending U into U'. Then $(I(G), <)$ is a directed set. All quotients $U \to U/G$ with induced maps by inclusions form an inductive system, that is the algebraic approximation of the universal principal bundle $EG \to BG$ ([28], [6]).

An algebraic construction of classifying maps (for principal bundles over quasi-projective varieties) is given in Lemma 1.6 in [28], that will be used in the last section.

2.2. Mixed quotients. Let X be a G-variety. For any $U \in I(G)$, the diagonal action of G on $X \times U$, which is always a free action, gives a principal bundle $X \times U \to X \times_G U = (X \times U)/G$, and thus the equivariant projection $X \times U \to U$ serves the fibre bundle $X \times_G U \to U/G$ with fibre X. Roughly saying, the

universal fibre bundle $X \times_G EG \to BG$ is approximated by those mixed quotients $X \times_G U \to U/G$ for $U \in I(G)$ (Edidin-Graham [6]).

We attention to the fact that in general the mixed quotients $X \times_G U$ exists as *algebraic spaces* in the sense of Artin, not as schemes (Proposition 22 [6]). To avoid this technicality, we may think of the following cases: In the complex case $k = \mathbb{C}$, a separated algebraic space of finite type admits the corresponding analytic space (Corollary 1.6 in [1]) (besides, in our convention X is assumed to be separated, of finite type and (G-)embeddable, hence $X \times_G U$ is also as an algebraic space, thus as an analytic space). In the algebraic context, we assume that X is a quasi-projective scheme with a linearlized G-action. Then the mixed quotient $X \times_G U$ exists as a quasi-projective scheme (Proposition 23 in [6]). Note that this quasi-projective hypothesis covers rather many interesting cases.

In fact, we will later apply (ordinary) transformation C_* to those mixed quotients in the complex case (by appealing to the transcendental method) and also in the quasi-projective case over k of characteristic 0. Presumably Kennedy's formulation (for embeddable schemes) would be extendable into the context of (embeddable) algebraic spaces, then this kind restriction mentioned above would not be needed.

2.3. G-equivariant cohomology. For any pair U and U' ($\in I(G)$) so that $U < U'$, we let $\iota_{U,U'} : X \times_G U \to X \times_G U'$ denote the natural inclusion and $r_{U',U} := \iota^*_{U,U'} : H^*(X \times_G U') \to H^*(X \times_G U)$ the induced homomorphism. Then we have a projective system $\{H^*(X \times_G U), r_{U,U'}\}$ and *the i-th equivariant cohomology* of X is given as

$$H^i_G(X) = \varprojlim_{I(G)} H^i(X \times_G U).$$

The formal sum is denoted by $H^*_G(X) = \prod H^i_G(X) = \lim_{\leftarrow} H^*(X \times_G U)$. We also denote by $r_U : H^*_G(X) \to H^*(X \times_G U)$ the canonical projection for U. Note that in the algebraic context, the cohomology groups should be replaced by *operational Chow groups* ([9], [6]).

Let ξ be a G-equivariant vector bundle $E \to X$ (i.e., E, X are G-varieties, the projection is G-equivariant), then ξ induces a vector bundle $E \times_G U \to X \times_G U$, denoted by ξ_U. The projective limit of Chern classes $c(\xi_U)$ gives *the G-equivariant Chern class* of ξ, which is denoted by $c^G(\xi) \in H^*_G(X)$. In particular, when $X = \{pt\}$, an equivariant vector bundle is $V \to \{pt\}$ being V a representation. The Chern class is denoted by $c^G(V) \in H^*_G(pt) = H^*(BG)$. Besides, the pullback of $c^G(V)$ via the trivial equivariant morphism $X \to \{pt\}$ is denoted by the same notation: $c^G(V) \in H^*_G(X)$.

Given a G-morphism $f : X \to Y$, the pullback $f^*_G : H^i_G(Y) \to H^i_G(X)$ is defined in a natural way: $f^*_G(\{\alpha_U\}) := \{(f \times_G id)^* \alpha_U\}$.

2.4. G-equivariant homology. We repeat Edidin-Graham's definition of the equivariant homology (Chow group) (Proposition 1 in [6]) but in a suitable form for the later use (again we describe it in the complex context but it works over any case).

At first we define a sub-order $<_*$ on $I(G)$: For any two $U(= V - S)$ and $U'(= V' - S')$, we say that $U <_* U'$ if there is a representation V_1 so that $V \oplus V_1 = V'$ and $U \oplus V_1 \subset U'$. Note that if $U_1 < U_2$, then there is U' so that $U_1 <_* U'$ and $U_2 <_* U'$ (e.g., $U' = V_1 \oplus V_2 - S_1 \oplus S_2$).

Let X be a complex G-variety of $\dim X = n$ (equidimensional). To each $U = V - S$ with $\dim V = l$ and $\operatorname{codim} S = s$, we assign a *truncated* homology

$$H_{trunc}(X \times_G U) := \bigoplus_{2(n-s)<i\le 2n} H_{i+2(l-g)}(X \times_G U).$$

Note that the range of dimension in the direct sum depends on U (precisely, the dimensions of V and S). This notation is convenient for us because we shall later think of *total* homology classes (*total Chern classes*) rather than a distinguished i-th homology class.

For each pair $U <_* U'$ ($V' = V \oplus V_1$), the diagram $U \leftarrow U \oplus V_1 \rightarrow U'$ of projection and injection induces

$$\begin{array}{ccccc} p = p_{U,U\oplus V_1} & & X \times_G (U \oplus V_1) & & \iota_{U\oplus V_1,U'} = \iota \\ & \swarrow & & \searrow & \\ X \times_G U & & & & X \times_G U' \end{array} \qquad (*)$$

This diagram $(*)$ induces the following isomorphisms for $2(n-s) < i \le 2n$ (we denote $(i^*)^{-1}$ by ι_*, in abusing the notation):

$$\begin{array}{ccccc} & p^* & H_{i+2(l+k-g)}(X \times_G (U \oplus V_1)) & \iota_* := (\iota^*)^{-1} & \\ & \simeq \nearrow & & \searrow \simeq & \\ H_{i+2(l-g)}(X \times_G U) & & & & H_{i+2(l+k-g)}(X \times_G U') \end{array} \qquad (*_H)$$

This is because ι is an open embedding (so the pullback ι^* is defined) and its complement $X \times_G U' - X \times_G (U \oplus V_1)$ has the (complex) codimension $\ge s$ (hence ι^* is isomorphic). Also $p = p_{U,U\oplus V_1}$ is the projection of a vector bundle so it induces an isomorphism p^*. The composition of isomorphisms in $(*_H)$ define a graded homomorphism of truncated homology groups, whose degrees are shifted by $k = \dim U' - \dim U$, denoted by

$$\varphi_{U,U'} : \bigoplus_{2(n-s)<i} H_{i+2(l-g)}(X \times_G U) \to \bigoplus_{2(n-s')<i} H_{i+2(l+k-g)}(X \times_G U').$$

This makes an inductive system with respect to the directed set $(I(G), <_*)$, moreover with the original order $<$ [1]. We define *the i-th equivariant homology group* to be (the limit of) the shifted-dimensional component of the truncated homology

$$H_i^G(X) = H_{i+2(\dim U - g)}(X \times_G U) \quad \text{(for } U \text{ with } \operatorname{codim} S \text{ high enough).}$$

[1] For any pair $U_1 < U_2$ with respect to the original order, take U' so that $U_1, U_2 <_* U'$ (e.g., $U' = U_1 \oplus U_2$). Then both $\varphi_{U_1,U'}$ and $\varphi_{U_2,U'}$ are isomorphic at least in the range $i > 2n - 2s_1$ ($s_1 = \operatorname{codim} S_1$), and hence we can define a canonical injective homomorphism φ_{U_1,U_2} ($:= (\varphi_{U_2,U'})^{-1} \circ \varphi_{U_1,U'}$) from the truncated homologies of U_1 to the one of U_2, that is so-called the double filtration argument in [28], [6]. Therefore we don't take care of underlying orders, so we use $<_*$ basically.

Thus $H_i^G(X)$ is trivial for $i > 2n$ and possibly nontrivial for any negative i. The direct sum is denoted by

$$H_*^G(X) = \oplus H_i^G(X) = \varinjlim H_{trunc}(X \times_G U).$$

For each U, the identification map is denoted by $\varphi_U : H_{trunc}(X \times_G U) \to H_*^G(X)$.

Given a proper G-morphism $f : X \to Y$ between G-varieties, we have an induced homomorphism $f_*^G : H_*^G(X) \to H_*^G(Y)$ defined by $f_*^G(\varphi_U(c)) := \varphi_U((f \times_G id)_*(c))$ as the limit of $(f \times_G id)_* : H_{trunc}(X \times_G U) \to H_{trunc}(Y \times_G U)$. Any other expected functorial properties are also satisfied, see [6].

As an example we illustrate a most simplest case: $G = GL(1)$, $X = \{pt\}$ and a sequence in $I(G)$ that is $\{U_m = \mathbb{C}^{m+1} - \{0\}\}$ with the action of all weights -1:

$$\begin{array}{cccccc} & & & (U_m \times \mathbb{C})/GL(1) & = \mathbb{P}^{m+1} - \{pt\} & \\ \cdots \overset{p}{\swarrow} & \searrow \overset{p}{\swarrow} & & & \searrow \;\; \cdots & \\ \mathbb{P}^1 \subset \cdots \subset & \mathbb{P}^m & & \subset & \mathbb{P}^{m+1} & \subset \cdots \subset \mathbb{P}^\infty \sim BGL(1) \end{array}$$

Note that $n = \dim X = 0$, $g = \dim G = 1$ and

$$H_{trunc}(X \times_G U_m) = \bigoplus_{-2(m+1)<i} H_{i+2m}(\mathbb{P}^m) \simeq \mathbb{Z}^{m+1}.$$

The map $\varphi_{U_m,U_l} : \mathbb{Z}^{m+1} \to \mathbb{Z}^{l+1}$ is a natural inclusion and hence $H_i^{GL(1)}(pt) = \mathbb{Z}$ for nonpositive even number i, and trivial otherwise.

Roughly saying, the direct sum of fibres of p's over a point approximates "the tangent space $T_{\mathbb{P}^\infty}$". In fact there will appears somewhat "(inverse) Chern class factor of $T_{\mathbb{P}^\infty}$" in our definition of C_*^G given in §3 (Remark 3.3).

2.5. G-fundamental class and Poincaré duality. For any U, the fundamental cycle $[X \times_G U]$ tends to a unique element of $H_{2n}^G(X)$, denoted by $[X]_G$. This is called *the G-equivariant fundamental class* of X. Note that we can identify $H_{2j}^G(X) = H_{2(j+l)}^G(X \times V)$ for any representation V ($\dim V = l$) through the pullback isomorphisms induced by $(X \times V) \times_G U \to X \times_G U$. Let W be an $(j+l)$-dimensional G-invariant reduced closed subscheme of $X \times V$ and $i : W \to X \times V$ the G-inclusion. Then W represents an equivariant homology class of X as $i_*^G([W]_G) \in H_{2(j+l)}^G(X \times V) = H_{2j}^G(X)$. *An j-dimensional G-equivariant algebraic cycle class* means a finite sum $\sum_k a_k\,(i_k)_*([W_k]_G) \in H_{2j}^G(X)$, where each W_k is an $(j+l)$-dimensional G-invariant subvariety of some $X \times V$. Here j is possibly negative ($-l \le j \le n$). Of course, a G-invariant cycle of X represents an equivariant cycle class of nonnegative dimension.

There is a well-defined homomorphism

$$\frown [X]_G : H_G^{2n-i}(X) \to H_i^G(X), \quad a \mapsto \varphi_U(r_U(a) \frown [X \times_G U]).$$

If X is nonsingular, this is isomorphic for each i, called *the G-equivariant Poincaré dual*. In particular, when X is a point, $H_{-k}^G(pt) \simeq H_G^k(pt) = H^k(BG)$. We denote by $Dual_G$ the inverse of the map $\frown [X]_G$ (for each i). The composite map

$r_U \circ Dual_G \circ \varphi_U$ coincides with the ordinary Poincaré dual of $X \times_G U$ on the truncated homology.

2.6. G-equivariant constructible functions. A constructible function over a complex algebraic variety X is an integer valued function $\alpha : X \to \mathbb{Z}$ which has a finite partition of X into constructible subsets so that the value of α is constant over each of the constructible sets. We let $\mathcal{F}(X)$ denote the Abelian group consisting of all constructible functions over X. Any constructible function $\alpha \in \mathcal{F}(X)$ is represented by $\alpha = \sum_{i=1}^{k} a_i 1\!\!1_{W_i}$ for some integers a_i and subvarieties W_i of X. Here $1\!\!1_W$ denotes the function taking values 1 for $x \in W$ and 0 otherwise. For any proper morphism $f : X \to Y$, we define the pushforward $f_* : \mathcal{F}(X) \to \mathcal{F}(Y)$ by

$$f(\alpha)(y) := \sum_{i=1}^{k} a_i\, \chi(f^{-1}(y) \cap W_i) \quad (\alpha = \sum_{i=1}^{k} a_i 1\!\!1_{W_i}, y \in Y),$$

where χ is the topological Euler characteristic with respect to the Borel-Moore homology groups. Note that $(f \circ g)_* = f_* \circ g_*$. Even if f is not proper, the sum in the right hand side may be finite for any α and y, and in that case we also denote the map by f_*. For a constructible function $\alpha \in \mathcal{F}(X)$, we define *the integral of α over X* (or say, the Euler characteristics of α) to be the value $f_*(\alpha) \in \mathcal{F}(pt) = \mathbb{Z}$ with $f : X \to \{pt\}$, that is

$$\int_X \alpha := f_*(\alpha) = \sum_{i=1}^{k} a_i\, \chi(W_i) \ \in \ \mathbb{Z}.$$

For any morphism $f : X \to Y$, the pullback $f^* : \mathcal{F}(Y) \to \mathcal{F}(X)$ is defined to be $f^*(\beta) := \beta \circ f$.

Remark 2.1. *In the case of the base field k of characteristic 0, the above definition of pushforward should be appropriately changed in terms of Lagrangian cycles, see* [16]. *In abusing the notation, we may use the letter $\chi(X) := \int_X 1\!\!1_X$ in this context too.*

Now let X be a variety with a G-action. The subgroup of $\mathcal{F}(X)$ consisting of G-invariant constructible functions is denoted by

$$\mathcal{F}^G_{inv}(X) := \{\ \alpha \in \mathcal{F}(X) \mid \alpha(g(x)) = \alpha(x),\, (x \in X, g \in G)\ \}.$$

For any $U <_* U'$ ($V' = V \oplus V_1$, $U = V - S$, $U' = V' - S'$), let $p : V' \to V$ be the projection to the first factor, then it induces a pullback homomorphism

$$\phi_{U,U'} := p^* : \mathcal{F}^G_{inv}(X \times V) \to \mathcal{F}^G_{inv}(X \times V'), \qquad \alpha \mapsto \alpha \circ (id \times p)$$

(we sometimes denote it by $\phi_{V,V'}$). Then, $\{\mathcal{F}^G_{inv}(X \times V), \phi_{U,U'}\}$ makes an inductive system, so we define

$$\mathcal{F}^G(X) := \varinjlim_{I(G)} \mathcal{F}^G_{inv}(X \times V).$$

An element of this limit group is called *a G-equivariant constructible functions associated to X*. The limit map is denoted by $\phi_U : \mathcal{F}^G_{inv}(X \times V) \to \mathcal{F}^G(X)$ (sometimes by ϕ_V).

In an obvious way, any G-invariant function over X is lifted to an invariant function over $X \times V$, and hence there is a canonical inclusion, denoted by ϕ_0,

$$\mathcal{F}^G_{inv}(X) \subset \mathcal{F}^G(X), \quad \alpha \mapsto \phi_0(\alpha) = \phi_V(\alpha \times 1\!\!1_V).$$

Note that if X is a point, then $\mathcal{F}^G_{inv}(pt) = \mathcal{F}(pt) \simeq \mathbb{Z}$ (consisting of constant functions) but $\mathcal{F}^G(pt)$ contains a lot of other invariant functions over representations V's.

For a proper G-morphism $X \to Y$, we define *the equivariant pushforward* homomorphism

$$f^G_* : \mathcal{F}^G(X) \to \mathcal{F}^G(Y), \quad f^G_*(\phi_U \alpha_U) := \phi_U((f \times id)_*(\alpha_U)),$$

that is the limit map of $(f \times id)_* : \mathcal{F}^G_{inv}(X \times V) \to \mathcal{F}^G_{inv}(Y \times V)$. It is easily checked that $(f \circ g)^G_* = f^G_* \circ g^G_*$ for proper G-morphisms. Similarly *the equivariant pullback* $f^*_G : \mathcal{F}^G(Y) \to \mathcal{F}^G(X)$ is also defined.

For any $\alpha \in \mathcal{F}^G(X)$, we define *the G-integral* of α to be $f^G_*(\alpha) \in \mathcal{F}^G(pt)$ by the pointed map $f : X \to \{pt\}$. So a G-integral is an equivariant constructible function over a point, not constant in general. In particular, if $\alpha \in \mathcal{F}^G_{inv}(X)$, more precisely, $\alpha = \phi_V(\alpha_0 \times 1\!\!1_V)$ for some $\alpha_0 \in \mathcal{F}^G_{inv}(X)$, then the G-integral of α is the constant $\int_X \alpha_0$:

$$\begin{aligned} f^G_*(\alpha) &= f^G_*(\phi_V(\alpha_0 \times 1\!\!1_V)) = \phi_V((f \times id)_*(\alpha_0 \times 1\!\!1_V)) \\ &= \phi_V(f_*(\alpha_0) \times 1\!\!1_V) = f_*(\alpha_0). \end{aligned}$$

Remark 2.2. *Instead of $\phi_{U,U'} = p^*$, we think of $\tilde\phi_{U,U'}$, the restriction of $\phi_{U,U'}$ to $\mathcal{F}^G_{inv}(X \times U)$, which is regarded as the composed map*

$$\begin{tikzcd}[column sep=small] & \mathcal{F}^G_{inv}(X \times (U \oplus V_1)) \arrow[dr, "(\iota_{U\oplus V_1,U'})_* = \iota_*"] & \\ \mathcal{F}^G_{inv}(X \times U) \arrow[ur, "p^* = (p_{U,U\oplus V_1})^*"] & & \mathcal{F}^G_{inv}(X \times U') \end{tikzcd} \qquad (*_F)$$

(ι is not proper but an open embedding, so ι_* is defined). Note that $\{\mathcal{F}^G_{inv}(X \times U), \tilde\phi_{U,U'}\}$ also makes an inductive system, but the limit group differs slightly from the above $\mathcal{F}^G(X)$. Later we will consider essentially this smaller limit group. The point is that $\tilde\phi_{U,U'}$ for constructible functions and $\varphi_{U,U'}$ for homology can be read off as topological Radon transforms, that is, roughly saying, pulling back and then pushing forward.

In [7] the authors studied the category whose objects are (nonsingular) varieties and "morphisms" of X to Y are diagrams $X \xleftarrow{p} M \xrightarrow{q} Y$ with p being a smooth morphism (the case of singular varieties is supported by [30]). For this category, the covariant functors q_*p^* of constructible functions and "twisted" q_*p^*

of homologies are defined, and the MacPherson-type natural transformation between these two functors is constructed. In the next section we will construct C_*^G throughout applying this construction to $\tilde{\phi}_{U,U'}$ and $\varphi_{U,U'}$.

3. Equivariant natural transformation

In this section we prove Theorem 1.1 in both contexts of the complex case and the quasi-projective case of characteristic 0 (see subsection 2.2). In the latter case, homologies should be read off as Chow homologies A_*.

3.1. Construction of C_*^G. For each $U = V - S \in I(G)$ which is non-empty, the inclusion $U \subset V$ is denoted by j_U and it induces $j_U^* : \mathcal{F}_{inv}^G(X \times V) \to \mathcal{F}_{inv}^G(X \times U)$. Since G acts freely on $X \times U$ (hence $X \times U \to X \times_G U$ is a principal bundle), any G-invariant reduced subscheme W of X has a principal quotient $W \to W/G$. Thus, to $1\!\!1_W \in \mathcal{F}_{inv}^G(X \times U)$ we assign $1\!\!1_{W/G} \in \mathcal{F}(X \times_G U)$, that actually makes an isomorphism of groups: so we identify $\mathcal{F}_{inv}^G(X \times U) = \mathcal{F}(X \times_G U)$.

As noted in the subsection 2.2, we can apply the (ordinary) MacPherson transformation to the mixed quotient $X \times_G U$:

$$C_* : \mathcal{F}(X \times_G U) \to H_*(X \times_G U).$$

We denote by TU_G for short, the vector bundle

$$X \times_G TU (= X \times_G (U \oplus V)) \to X \times_G U$$

and its Chern class by $c(TU_G) \in H^*(X \times_G U)$. That is, $c(TU_G) := r_U c^G(V)$, where $r_U : H_G^*(X) \to H^*(X \times_G U)$ is the canonical projection and $c^G(V)$ is the Chern class of the representation V. Combining the above maps, we define

$$T_{U,*} = c(TU_G)^{-1} \frown C_* \circ j_U^* : \mathcal{F}_{inv}^G(X \times V) \to H_*(X \times_G U).$$

Its projection to the truncated homology is also denoted by the same letter.

Lemma 3.1. *For $U <_* U'$, the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{F}_{inv}^G(X \times V) & \xrightarrow{T_{U,*}} & H_{trunc}(X \times_G U) \\ \phi_{U,U'} \downarrow & & \downarrow \varphi_{U,U'} \\ \mathcal{F}_{inv}^G(X \times V') & \xrightarrow{T_{U',*}} & H_{trunc}(X \times_G U') \end{array}$$

Proof: : We write $T_{U,*} = C_{U,*} \circ j_U^*$ ($C_{U,*} := c(TU_G)^{-1} \frown C_*$) for short. For $U <_* U'$ ($V' = V \oplus V_1$, $U = U - S$, $U' = V' - S'$), we take the following diagram, in which the left vertical map p^* is $\phi_{U,U'}$, the middle vertical map $\iota_* \circ p^*$ is $\tilde{\phi}_{U,U'}$

and the right vertical map $\iota_* \circ p^*$ is $\varphi_{U,U'}$:

$$\begin{array}{ccccc}
\mathcal{F}^G_{inv}(X \times V) & \xrightarrow{j^*_U} & \mathcal{F}(X \times_G U) & \xrightarrow{C_{U,*}} & H_{trunc}(X \times_G U) \\
\downarrow\ p^* & (1) & \downarrow\ p^* & (2) & \downarrow\ p^* \\
\mathcal{F}^G_{inv}(X \times V') & \xrightarrow{j^*_{U\oplus V_1}} & \mathcal{F}(X \times_G (U \oplus V_1)) & \xrightarrow{C_{U\oplus V_1,*}} & H_{trunc}(X \times_G (U \oplus V_1)) \\
\| & (3) & \downarrow\ \iota_* & (4) & \downarrow\ \iota_* \\
\mathcal{F}^G_{inv}(X \times V') & \xrightarrow[j^*_{U'}]{} & \mathcal{F}(X \times_G U') & \xrightarrow[C_{U',*}]{} & H_{trunc}(X \times_G U')
\end{array}$$

We show that maps of the big square surrounding the diagram commutes. First, since $\tilde{\phi}_{U,U'}$ is a restriction of $\phi_{U,U'}$ as noted in Remark 2.2, the left half of the big square "(1) +(3)" (forgetting the middle arrow $j^*_{U\oplus V_1}$) commutes, although (1) commutes but (3) does not. Next look at (2). For the projection $p = p_{U,U\oplus V_1}$: $X \times_G (U \oplus V_1) \to X \times_G U$, we let $c(p)$ denote the Chern class of relative tangents of p, i.e., $c(p) = r_{U\oplus V_1} c^G(V_1) \in H^*(X \times_G (U \oplus V_1))$. It then follows from the Verdier-Riemann-Roch formula ([10], [30]) that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{F}(X \times_G U) & \xrightarrow{C_*} & H_*(X \times_G U) \\
p^* \downarrow & & \downarrow c(p) \frown p^* \\
\mathcal{F}(X \times_G (U \oplus V_1)) & \xrightarrow{C_*} & H_*(X \times_G (U \oplus V_1))
\end{array}$$

Then for $\alpha \in \mathcal{F}(X \times_G U)$ we have

$$\begin{aligned}
p^* C_{U,*}(\alpha) &= p^* \left(c(TU_G)^{-1} \frown C_*(\alpha) \right) \\
&= p^* c(TU_G)^{-1} \frown p^* C_*(\alpha) \\
&= p^* c(TU_G)^{-1} r_{U\oplus V_1} c^G(V_1)^{-1} \left(r_{U\oplus V_1} c^G(V_1) \frown p^* C_*(\alpha) \right) \\
&= c(T(U \oplus V_1)_G)^{-1} \frown (c(p) \frown p^* C_*(\alpha)) \\
&= c(T(U \oplus V_1)_G)^{-1} \frown C_*(p^*(\alpha)) \\
&= C_{U\oplus V_1,*}(p^* \alpha).
\end{aligned}$$

So the square (2) commutes.

The remaining part is (4). The inclusion $\iota : U \oplus V_1 \to U'$ is not proper so we don't use the functoriality of C_*, but we recall that its complement $U' - U \oplus V_1$ has a sufficiently large (complex) codimension $\geq s$ ($s = \operatorname{codim} S$ of U). Therefore, for any $1\!\!1_W \in \mathcal{F}(X \times_G (U \oplus V_1))$, the difference between $c = \iota_* C_*(1\!\!1_W)$ and $c' = C_*(\iota_* 1\!\!1_W)$ has support with codim $\geq s$. Since $c(T(U \oplus V_1)_G) = \iota^* c(TU'_G)$ is obvious, we have

$$\iota_* C_{U\oplus V_1,*}(1\!\!1_W) = c(TU'_G)^{-1} \frown c \equiv c(TU'_G)^{-1} \frown c' = C_{U',*}(\iota_* 1\!\!1_W)$$

up to the truncated part (of the dimension $> 2(n + l' - g) - 2s$). Thus (4) commutes. This completes the proof.

Definition 3.2. *We define the limit homomorphism*

$$C_*^G := \varinjlim T_{U,*} : \mathcal{F}^G(X) \to H_*^G(X), \qquad \phi_U(\alpha_U) \mapsto \varphi_U \circ T_{U,*}(\alpha_U).$$

where $\phi_U : \mathcal{F}^G_{inv}(X \times V) \to \mathcal{F}^G(X)$ *and* $\varphi_U : H_{trunc}(X \times_G U) \to H_*^G(X)$ *are identification maps.*

3.2. Proofs of Theorem 1.1. The rest is to show the following properties:

(a) $C_*^G : \mathcal{F}^G(X) \to H_*^G(X)$ satisfies the expected naturality:
  (i) $C_*^G(\alpha+\beta) = C_*^G(\alpha) + C_*^G(\beta)$, $\alpha, \beta \in \mathcal{F}^G(X)$;
  (ii) $C_*^G \circ f_*^G = f_*^G \circ C_*^G$ for proper G-morphisms $f : X \to Y$;
  (iii) If X is non-singular, then $C_*^G(1\!\!1_X^G) = c_*^G(TX) \frown [X]_G$.

(b) Suppose that for each U we are given a homomorphism $DT_{U,*} : \mathcal{F}^G_{inv}(X \times V) \to H_{trunc}(X \times_G U)$ commuting with the structure homomorphisms ($\phi_{U,U'}$ and $\varphi_{U,U'}$) such that its inductive limit $D_*^G : \mathcal{F}^G \to H_*^G$ satisfying the above properties (i), (ii) and (iii). Then D_*^G coincides with our C_*^G.

Proof: : (a): (i) is trivial. (ii) follows from the fact that $C_* \circ (f \times_G id)_* = (f \times_G id)_* \circ C_*$ for the induced proper map $f \times_G id : X \times_G U \to Y \times_G U$. To show (iii) we recall the normalization property of C_*. If X is nonsingular, then $X \times_G U$ is also, hence $C_*(1\!\!1_{X \times_G U}) = c(T(X \times_G U)) \frown [X \times_G U]$. Since $c(T(X \times_G U)) = c(TX_G)c(TU_G)$ where $c(TX_G) = r_U c^G(TX)$, we have

$$c(TU_G)^{-1} \frown C_*(1\!\!1_{X \times_G U}) = r_U c^G(TX) \frown [X \times_G U].$$

Thus, by definitions,

$$\begin{aligned} C_*^G(1\!\!1_X) &= C_*^G(\phi_U 1\!\!1_{X \times V}) = \varphi_U \circ T_{U,*}(1\!\!1_{X \times V}) \\ &= \varphi_U \left(c(TU_G)^{-1} \frown C_*(1\!\!1_{X \times_G U}) \right) \\ &= \varphi_U \left(r_U c^G(TX) \frown [X \times_G U] \right) \\ &= c^G(TX) \frown [X]_G. \end{aligned}$$

(b): This is straightforward from the uniqueness of ordinary C_*: By (iii) it turns out that $DT_{U,*}$ also have the natural functoriality and the normalization condition up to the truncated homologies. Hence by the uniqueness of C_*, $DT_{U,*}$ must coincide with our $T_{U,*}$ (up to the truncated homologies). Since we can take $U (= V - S)$ so that $\operatorname{codim} S$ is any large number, thus $D_*^G = C_*^G$.

This completes the proof of Theorem 1.1.

3.3. Remark on limit systems. It is possible to take some different inductive systems for the definition of equivariant (co)homology. We may replace the structure homomorphisms $r_{U',U} : H^*(X \times_G U') \to H^*(X \times_G U)$ and $\varphi_{U,U'} : H_{trunc}(X \times_G U) \to H_{trunc}(X \times_G U)$ for $U <_* U'$ ($V' = V \oplus V_1$) by

$$\tilde{r}_{U',U} := r_{U'} c^G(V_1) \cdot r_{U',U}, \qquad \tilde{\varphi}_{U,U'} := r_{U'} c^G(V_1) \frown \varphi_{U,U'}.$$

Let $h_G^*(X)$ and $h_*^G(X)$ denote the limits of these "twisted" systems, in a moment. This definition of $h_*^G(X)$ is very similar to the construction of a certain homology theory of a provariety (Yokura [31]).

We define $c(T_{BG}) \in h_G^*(X)$ to be the projective limit $\{c(TU_G)\}_U$, which is well-defined:

$$\tilde{r}_{U',U}c(TU_G) = r_{U'}c^G(V_1) \cdot r_{U',U}c(TU_G) = r_{U'}c^G(V_1) \cdot r_{U'}c^G(V) = c(TU'_G).$$

In fact $c(T_{BG})$ corresponds to the identity element $1 \in H_G^*(X)$. Further we have a group isomorphism $c^{-1} : h_*^G(X) \to H_*^G(X)$ defined by an automorphism of $H_{trunc}(X \times_G U)$, $\xi \mapsto c(TU_G)^{-1} \frown \xi$. Then, the above construction of C_*^G is factored as follows:

$$\mathcal{F}^G(X) \xrightarrow{\lim C_*} h_*^G(X) \xrightarrow{c^{-1}} H_*^G(X).$$

The first map is just the inductive limit of (ordinary) MacPherson transforms $(C_*)_U$, which is well-defined by the VRR (this is actually discussed in [31] in a more general setting). The second map corrects the twisting of the limit systems, that roughly means reducing the Chern class factor of "horizontal tangents π^*T_{BG}" of the fibration $\pi : X \times_G EG \to BG$.

3.4. MacPherson's transformation for quotient stacks. This subsection is a bit isolated from others. We work here over k of characteristic 0 with use of A_* and assume again G-schemes X to be quai-projective with linearlized G-action. "The quotient of X via G" exists in the category of algebraic stacks, i.e., the quotient stack $\mathcal{X} = [X/G]$ associated to the groupoid $G \times X \to X \times X$ ([29] Appendix, [6]). Then $\mathcal{X}$ is a category itself, whose objects are principal G-bundles $E \to T$ together with G-equivariant morphism $E \to X$ and its arrows are morphisms between principal bundles which make the equivariant morphisms to X commute.

In [6] (Proposition 16) , *the integral Chow groups of a quotient stack* $\mathcal{X} = [X/G]$ is introduced as $\bar{A}_*(\mathcal{X}) := A_{*-g}^G(X)$ where $g = \dim G$, which is independent from the choice of the presentation (to avoid any confusion, we put a bar over the letter A_*). The following lemma is shown in the same manner:

Lemma 3.3. *If* $[X/G] \simeq [Y/H]$ *as quotient stacks, then* $\mathcal{F}_{inv}^G(X) \simeq \mathcal{F}_{inv}^H(Y)$.

Proof: Let $\mathcal{X} := [X/G]$. Since the diagonal of a quotient stack is representative, the fibre product $X \times_{\mathcal{X}} Y$ is a scheme. It has an obvious action of $G \times H$. The diagram $X \overset{p_1}{\leftarrow} X \times_{\mathcal{X}} Y \overset{p_2}{\to} Y$ of G and H-equivariant projections is regarded as the object corresponding to the morphism $Y \to \mathcal{X}$ (p_2 being the principal G-bundle), as well the object corresponding to $X \to \mathcal{X}(\simeq [Y/H])$ (p_1 being the principal H-bundle). Then via the pullbacks of constructible functions, we have isomorphisms $\mathcal{F}_{inv}^G(X) \overset{p_1^*}{\to} \mathcal{F}_{inv}^{G\times H}(X \times_{\mathcal{X}} Y) \overset{p_2^*}{\leftarrow} \mathcal{F}_{inv}^H(Y)$.

Thus we define the Abelian group of constructible functions for $\mathcal{X} = [X/G]$ to be $\bar{\mathcal{F}}_{inv}(\mathcal{X}) := \mathcal{F}_{inv}^G(X)$.

Remark 3.4. *If G acts on X trivially, $\bar{A}_*(\mathcal{X})$ is identified with $A_*(X)$. As in* [6]*, if the action of G is (locally) proper, then $\mathcal{X}$ becomes an algebraic space so the Chow group $A_*(\mathcal{X})$ makes sense, and then Theorem 3 in* [6] *says that $A_*(\mathcal{X}) \otimes \mathbb{Q} \simeq A^G_{*-g}(X) \otimes \mathbb{Q}\, (= \bar{A}_*(\mathcal{X}) \otimes \mathbb{Q})$.*

Let us consider the following category: the objects are quotient stacks $[X/G]$ of quasi-projective varieties X with linearlized actions of some algebraic groups G, the arrows are morphisms $\bar{f} : [X/G] \to [Y/H]$, $f : X \to Y$ being quasi-projective.

It is easily verified that for this category of quotients, $\bar{\mathcal{F}}$ and $\bar{A}_*$ are covariant functors. It follows from Theorem 1.1 that

Theorem 3.5. *For the category of quotients $\mathcal{X}$ having presentations $[X/G]$ as above, there is a natural transformation $C_* : \bar{\mathcal{F}}_{inv}(\mathcal{X}) \to \bar{A}_*(\mathcal{X})$ so that for any nonsingular varieties $\mathcal{X} = X$ (with trivial actions), it holds that $C_*(1\!\!1_X) = c(TX) \frown [X]$.*

It is also possible to define $\mathcal{F}([X/G])$ as the inductive limit of $\mathcal{F}_{inv}([X \times V/G])$ and extend C_* to the group (cf. Corollary 4.3 below).

As noted in 2.2, presumably the above theorem would be stated for general quotient stacks (i.e., without quasi-projectiveness) by generalizing Kennedy's formulation to algebraic spaces. Not only for quotient stacks, also for (general) algebraic stacks, MacPherson's transformation C_* is expected, but the author does not know how to manage it.

4. Some properties of C^G_*

4.1. Components of $C^G_*(1\!\!1_X)$. It is clear by definition that $C^G_*(1\!\!1_X)$ consists of $C^G_i(1\!\!1_X) \in H^G_{2i}(X)$, which is called *the i-th component*, furthermore $C^G_i(1\!\!1_X)$ is always trivial for any negative i (e.g., see Corollary 5.2).

We remark about the lowest 0-th component and the top n-th one of $C^G_*(1\!\!1_X)$ for a projective G-variety X of (equi)dimension n.

Let $f : X \to \{pt\}$, the pointed map. For any invariant constructible function $\alpha \in \mathcal{F}^G_{inv}(X)$ *the degree* of the 0-th component $G_0(\alpha)$ is defined to be the number $f^G_* C^G_0(\alpha) \in \mathcal{F}^G_{inv}(pt) = \mathbb{Z}$. It is easily seen that

$$\text{the degree of } C^G_0(\alpha) = \int_X \alpha \ \ \in \ \mathbb{Z}.$$

In fact, for $\alpha = 1\!\!1_X$,

$$f^G_* C^G_0(1\!\!1_X) \overset{(1)}{=} C^G_0 f^G_*(1\!\!1_X) \overset{(2)}{=} \chi(X) C^G_0(1\!\!1_{pt}) \overset{(3)}{=} \chi(X)\ [pt]_G.$$

Here (1) is by the naturality, (2) comes from the linearity and the fact that $f^G_*(1\!\!1_X) = \int_X 1\!\!1_X = \chi(X)$ as seen before, and (3) is the normalization condition $C^G_*(1\!\!1_{pt}) = c^G(Tpt) \frown [pt]_G = [pt]_G$ (corresponding to $1 \in H^*(BG)$).

As to the top component, recall the ordinary case: $C_n(1\!\!1_X) = [X] \in H_{2n}(X)$. In our equivariant setting, $C^G_*(1\!\!1_X)$ is the limit of $c(TU_G)^{-1} \frown C_*(1\!\!1_{X \times_G U})$ whose top component is the fundamental class $[X \times_G U]$, thus $C^G_n(1\!\!1_X) = [X]_G$.

4.2. Change of groups. Let G_0 be the closed subgroup of G, and X a G_0-variety of dimension n. Then $X \times_{G_0} G$ becomes a G-variety of dimension $n+k$ ($k = \dim G/G_0$). The G-constructible functions and equivariant homology of the mixed space are identified as follows:

Proposition 4.1. *There are canonical isomorphisms so that the following diagram commutes:*

$$\begin{array}{ccccc} \mathcal{F}^G(X \times_{G_0} G) & \xrightarrow{C_*^G} & H^G_{2(n+k)-*}(X \times_{G_0} G) & \xleftarrow{\cap fund} & H^*_G(X \times_{G_0} G) \\ \simeq\downarrow & & \downarrow\simeq & & \downarrow\simeq \\ \mathcal{F}^{G_0}(X) & \xrightarrow{C_*^{G_0}} & H^{G_0}_{2n-*}(X) & \xleftarrow{\cap fund} & H^*_{G_0}(X) \end{array}$$

Proof: Take $U \in I(G)$, an open set of a representation V of G over which the action is free. Of course, G_0 acts freely on U, i.e., $U \in I(G_0)$. We denote simply a point of $(X \times_{G_0} G) \times_G U$ by $[[x,a],u]$ so that $[[x,a],u] = [[h.x, ha], u] = [[h.x, hag^{-1}], g.u]$ for $h \in G_0$ and $g \in G$. Then, to each $[[x,1],u]$, we assign $[x,u]$ to get a (well-defined) isomorphism $(X \times_{G_0} G) \times_G U \xrightarrow{\sim} X \times_{G_0} U$. Also we can see an one-to-one correspondence between invariant subvarieties of $(X \times_{G_0} G) \times V$ and the one of $X \times V$. By these identifications and the construction of C_*^G, the claim follows.

4.3. Cross products. In (ordinary) MacPherson's theory, the cross product formula is known ([18]): for $\alpha \in \mathcal{F}(X)$ and $\beta \in \mathcal{F}(Y)$, $\alpha \times \beta \in \mathcal{F}(X \times Y)$ is defined to be $\alpha \times \beta(x,y) := \alpha(x) \cdot \beta(y)$, and it then holds that $C_*(\alpha \times \beta) = C_*(\alpha) \times C_*(\beta)$ where $\times$ in the right hand side means the homology cross product.

For G-varieties X and Y, the cross product

$$H^G_i(X) \otimes H^G_j(Y) \to H^G_{i+j}(X \times Y), \quad (\xi, \xi') \mapsto \xi \times \xi'$$

is well-defined (Def-Prop. 2 of [6]). In fact, for any $U = V - S$ and $U' = V' - S'$ in $I(G)$, we set the isomorphism $s : (X \times U) \times (Y \times U') \to X \times Y \times (U \oplus U')$ by $(x, u, y, u') \mapsto (x, y, u \oplus u')$ (also the isomorphism between the quotients is denoted by s), and then the limit of

$$H_{trunc}(X \times_G U) \oplus H_{trunc}(Y \times_G U') \to H_{trunc}((X \times Y) \times_G (U \oplus U')),$$

$(c, c') \mapsto s_*(c \times c')$, gives the above cross product. In a similar way, we define the exterior product of G-constructible functions,

$$\mathcal{F}^G_{inv}(X \times U) \otimes \mathcal{F}^G_{inv}(Y \times U') \to \mathcal{F}^G_{inv}((X \times Y) \times_G (U \oplus U')),$$

$(\alpha, \beta) \mapsto s_*(\alpha \times \beta)$. By the (ordinary) theory, we see that $C_* s_*(\alpha \times \beta) = s_* C_*(\alpha \times \beta) = s_*(C_*(\alpha) \times C_*(\beta))$. Multiplying both sides by inverse Chern class factors ($c(TU_G)^{-1}$ etc) and then taking the limits, we have

$$C_*^G(\alpha \times \beta) = C_*^G(\alpha) \times C_*^G(\beta) \quad (\alpha \in \mathcal{F}^G(X),\ \beta \in \mathcal{F}^G(Y)).$$

4.4. Equivariant Verdier-Riemann-Roch. We can formulate an equivariant version of the VRR formula for smooth morphisms ([30]).

Theorem 4.2. *Let $f : X \to Y$ be a G-equivariant smooth morphism with a G-equivariant relative tangent bundle ν of dimension m. Then the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{F}^G(Y) & \xrightarrow{C_*^G} & H_*^G(Y) \\ f^* \downarrow & & \downarrow f^{**} \\ \mathcal{F}^G(X) & \xrightarrow{C_*^G} & H_{*+2m}^G(X) \end{array}$$

where f^{**} is defined by $f^{**}(c) := c^G(\nu) \frown f^*(c)$.

Proof: Given a $f : X \to Y$ as above, then for any U, we set $f' := f \times_G id : X \times_G U \to Y \times_G U$, which is an smooth morphism with the relative bundle $\nu_G := \nu \times_G U$ over $X \times_G U$. We apply the ordinary VRR formula to this f', and then we have

$$\begin{array}{ccc} \mathcal{F}(Y \times_G U) & \xrightarrow{C_{U,*}} & H_*(Y \times_G U) \\ (f')^* \downarrow & & \downarrow (f')^{**} \\ \mathcal{F}(X \times_G U) & \xrightarrow{C_{U,*}} & H_{*+2m}(X \times_G U) \end{array}$$

where $C_{U,*} = c(TU_G)^{-1} \frown C_*$ and $(f')^{**} = c(\nu_G) \frown (f')^*$. This commutative diagram for U and the one for U' with $U <_* U'$ are compatible with $\tilde{\phi}_{U,U'}$ and $\varphi_{U,U'}$, so we see the claim similarly as in the proof of Lemma 3.1.

Corollary 4.3. *The limit homomorphism*

$$\varinjlim (C_*^G \circ \phi_0) : \mathcal{F}^G(X) = \varinjlim \mathcal{F}_{inv}^G(X \times V) \to \varinjlim H_*^G(X \times V)$$

coincides with the composition $p^{**} \circ C_*^G$ of the transformation $C_*^G : \mathcal{F}^G(X) \to H_*^G(X)$ and the twisted identification $p^{**} : H_*^G(X) \xrightarrow{\simeq} H_{*+2\dim V}^G(X \times V)$, $p^{**} = c^G(V) \frown p^*$, where p is the projection $X \times V \to X$.

Proof: The claim is shown by the following diagram whose right square is a simple VRR

$$\begin{array}{ccccc} \mathcal{F}_{inv}^G((X \times V) \times V) & \xrightarrow{\phi_V} & \mathcal{F}^G(X \times V) & \xrightarrow{C_*^G} & H_{*+2l}^G(X \times V) \\ \uparrow (p \times id)^* & & \uparrow p^* & & \simeq \uparrow p^{**} \\ \mathcal{F}_{inv}^G(X \times V) & \xrightarrow[\phi_V]{} & \mathcal{F}^G(X) & \xrightarrow[C_*^G]{} & H_*^G(X) \end{array}$$

where first two ϕ_V's are identification maps of the inductive limits. Clearly, $\phi_V \circ (p \times id)^*$ equals the canonical inclusion $\phi_0 : \mathcal{F}_{inv}^G(X \times V) \subset \mathcal{F}^G(X \times V)$.

5. Equivariant Chern-Mather class

5.1. Construction of C^M_G. First of all, we recall that there are two key factors in (ordinary) MacPherson's Chern class, *the Chern-Mather class* and *the local Euler obstruction*, see [19] for the detail. The local Euler obstruction Eu_X of a variety X is a constructible function of X assigning to $x \in X$ an local invariant of the germ (X, x), which is defined using obstruction theory (through the transcendental method) in [19], see also [12] for the purely algebraic definition. It defines an isomorphism $Eu : Z(X) \to \mathcal{F}(X)$ sending algebraic cycles $\sum n_k W_k$ to $\sum n_k Eu_{W_k}$, and C_* is constructed so that $C_*(\sum n_k Eu_{W_k}) = \sum n_k (i_k)_* C^M(W_k)$ where C^M means the Chern-Mather class, $i_k : W_k \to X$ is the inclusion map and $(i_k)_*$ is the induced homomorphism of homology groups.

Almost automatically, this relation among C_*, C^M and Eu is lifted into the equivariant version. Let X be a G-reduced scheme of equidimension n and assumed to be G-embeddable into some G-nonsingular variety, say M. As the same in the ordinary case without actions, we have *the G-equivariant Nash blow-up* of $\nu : \widehat{X} \to X$: Here $\widehat{X}$ is given as the closure of the regular part X_{Reg} in the Grassmanian $Gr_n(TM)$ of n-planes of TM on which G acts naturally, and $\nu : \widehat{X} \to X$ is the natural projection, which is an equivariant proper morphism.

$$\begin{array}{ccc} \widehat{X} & \subset & Gr_n(TM) \\ \nu \downarrow & & \downarrow \\ X & \subset & M \end{array}$$

Let $\widehat{TX}$ be the G-Nash tangent bundle over $\widehat{X}$, that is the restriction over $\widehat{X}$ of the tautological G-vector bundle of the Grassmanian. Then, we define *the G-equivariant Chern-Mather class of* X to be

$$C^M_G(X) := \nu^G_*(c^G(\widehat{TX}) \frown [\widehat{X}]_G) \in H^G_*(X).$$

Note that ν is actually made by the local embedding of X and the gluing process, so this construction is independent from the choice of the ambient space (cf. [19], [16]).

For a G-variety X, let $Z^G_{inv}(X)$ denote the subgroup of $Z(X)$ consisting of G-invariant algebraic cycles, and let $C^M_G : Z^G_{inv}(X) \to H^G_*(X)$ denote the map sending invariant cycles $\sum n_k W_k$ to $\sum n_k (i_k)^G_* C^M_G(W_k)$ where each W_k is a G-invariant reduced closed subscheme of X. We write the canonical inclusion by $\phi_0 : \mathcal{F}^G_{inv}(X) \subset \mathcal{F}^G(X)$.

Proposition 5.1. *Let X be a G-variety. Then,*
(1) Eu_X is G-invariant and $Eu : Z^G_{inv}(X) \to \mathcal{F}^G_{inv}(X)$ is an isomorphism;
(2) The following diagram commutes:

$$\begin{array}{ccccc} & Eu & Z^G_{inv}(X) & C^M_G & \\ & \overset{\simeq}{\swarrow} & & \searrow & \\ \mathcal{F}^G_{inv}(X) & & \underset{C^G_* \circ \phi_0}{\longrightarrow} & & H^G_*(X) \end{array}$$

Proof: (1) For any reduced G-invariant subscheme W and for any $g \in G$, $g : (W, x) \to (W, g(x))$ is isomorphic, so $Eu_W(x) = Eu_W(g(x))$, i.e., $Eu_W \in \mathcal{F}^G_{inv}(X)$. Since $Eu : Z(X) \to \mathcal{F}(X)$ is an isomorphism, its restriction to $Z^G_{inv}(X)$ is an isomorphism into $\mathcal{F}^G_{inv}(X)$. The surjectivity is shown similarly as in [19] by the induction of dimension using the fact that the singular loci of a G-invariant variety is also G-invariant.
(2) Let W be an invariant reduced subscheme of X. First, we see

$$Eu_W(x) = Eu_{W\times U}(x,u) = Eu_{W\times_G U}([x,u]).$$

This follows from three facts: $Eu_{W\times W'}(x,y) = Eu_W(x)Eu_{W'}(y)$, $Eu_W(x) = 1$ if (W,x) is nonsingular, and $W \times U \to W \times_G U$ is a principal G-bundle. Then, the map

$$j_U^* \circ \phi_{0,U} : \mathcal{F}^G_{inv}(X) \to \mathcal{F}^G_{inv}(X \times V) \to \mathcal{F}(X \times_G U)$$

sends Eu_W to $Eu(W \times_G U)$.

Second, let $\nu : \widehat{W} \to W$ be the G-Nash blow-up of W, and set $W_G := W \times_G U$. Since $W_G \to U/G$ is a bundle with fibre W over the smooth base space, it turns out that $\nu' = \nu \times_G id : \widehat{W} \times_G U \to W_G$ gives the (ordinary) Nash blow-up of W_G, i.e., $\widehat{W_G} = \widehat{W} \times_G U$. Then the (ordinary) Nash tangent bundle $\widehat{TW_G}$ splits to two factors coming from the G-Nash tangent bundle $\widehat{TW}$ and TU. By these facts and the fact that $C^M = C_* \circ Eu$ as mentioned, we have

$$\begin{aligned} C_* \circ j_U^* \circ \phi_{0,U}(Eu_W) &= C_* \circ Eu(W \times_G U) = C_* \circ Eu(W_G) \\ &= C^M(W_G) = (\nu')_*(c(\widehat{TW_G}) \frown [\widehat{W_G}]) \\ &= (\nu')_* c((\widehat{TW} \times_G U) \oplus (\widehat{W} \times_G TU)) \frown [\widehat{W_G}]) \\ &= c(TU_G) \frown ((\nu')_*(c(\widehat{TW} \times_G U) \frown [\widehat{W}_G])). \end{aligned}$$

Multiply both sides by $c(TU_G)^{-1}$ and take the limit, then we have $C^G_* \circ \phi_0(Eu_W) = C^M_G(W)$.

By its definition the Mather class $C^M_G(X)$ has no negative-dimensional component of $H^G_*(X)$. Thus it follows that

Corollary 5.2. *For any G-invariant constructible function $\alpha \in \mathcal{F}^G_{inv}(X)$, each negative-dimensional component $C^G_i(\alpha)$ $(i < 0)$ is trivial $(\in H^G_{2i}(X))$.*

This elementary fact is not very clear in the functorial definition of C^G_*. If we assume the existence of G-equivariant desingularizations of X being considered (or assume that the quotient X/G becomes a variety), then the above corollary immediately follows from functorial and normalization properties of C^G_*

Next, in the diagram of Proposition 5.1 let us replace X by $X \times V$ with a representation V and take the limit of groups. Then we have

Corollary 5.3. *Throughout the twisted identification via p^{**} as in Corollary 4.3, C_*^G is factored by the limit map of Eu^{-1} and the one of C_G^M, i.e., $p^{**} \circ C_*^G = \varinjlim C_G^M \circ \varinjlim Eu^{-1}$:*

$$
\begin{array}{ccccc}
 & & \varinjlim Z_{inv}^G(X \times V) & & \\
 & \simeq \nearrow & & \searrow & \\
\varinjlim \mathcal{F}_{inv}^G(X \times V) & & \longrightarrow & & \varinjlim H_{*+2l}^G(X \times V) \\
\| & & & & p^{**} \uparrow \simeq \\
\mathcal{F}^G(X) & & \underset{C_*^G}{\longrightarrow} & & H_*^G(X)
\end{array}
$$

Remark 5.4. *$C_G^M(X)$ is a kind of the Mather class with respect to the "fibrewise Nash-blowing up" of $X \times_G U \to U/G$ as seen in the proof of Proposition 5.1. This explains again the reason that $c(TU_G)^{-1}$ appears in the definition of $T_{U,*}$ and the twisted identification appears in Corollaries 4.3 and 5.3 (cf. Remark 3.3). We may have started to define the G-equivariant MacPherson transformation $C_*^G : \mathcal{F}_{inv}^G \to H_*^G$ by $C_G^M \circ Eu^{-1}$ as described above (or using a certain analogy to Lagrangian cycles), but it would become the same thing as we have done. To prove the functoriality of pushforwards would be harder than our simple argument.*

Remark 5.5. *As another story of Chern classes for singular spaces, there is so-called the Fulton canonical class $C^F(X)$ (Example 4.2.6 [9]): Let $s(X, M)$ denote the Segre covariant class of a closed subscheme X in a non-singular variety M, and then $C^F(X)$ is defined to be $c(TM|_X) \frown s(X, M)$. It turns out that $C^F(X)$ depends only on X, not on the choice of the ambient space M. In the equivariant setting, it is also possible to make the G-version of Fulton's class $C_G^F(X) \in H_*^G(X)$ for a G-scheme X using an equivariant blowing-up by the G-invariant ideal sheaf defining X in M. Then C_G^{SM}, C_G^M and C_G^F would be "basic" equivariant Chern homology classes like as the ordinary case, and they would have particular interests in singularity theory.*

5.2. Restriction: ordinary transformation without G-action. Take a fibre of $X \times_G U \to U/G$, and denote the inclusion of the fibre by $i : X \subset X \times_G U$ in abusing the notation. Then we have a homomorphism $i^* : H_*^G(X) \to H_*(X)$, that is independent from the choice of fibre. Through i^*, our C_*^G recovers the ordinary MacPherson transformation C_*.

Proposition 5.6. *We have the commutative diagram:*

$$
\begin{array}{ccccc}
\mathcal{F}_{inv}^G(X) & \overset{Eu^{-1}}{\to} & Z_{inv}^G(X) & \overset{C_G^M}{\to} & H_*^G(X) \\
\cap & & \cap & & \downarrow i^* \\
\mathcal{F}(X) & \overset{Eu^{-1}}{\to} & Z(X) & \overset{C^M}{\to} & H_*(X)
\end{array}
$$

Proof: The left square means the restriction of Eu^{-1} to $\mathcal{F}^G_{inv}(X)$, so it commutes. As seen in the proof of Proposition 5.1, the restriction of $\nu \times_G id : \widehat{X} \times_G U \to X \times_G U$ to a fibre $i(X)$ is isomorphic to the ordinary Nash blowing-up $\nu : \widehat{X} \to X$, thus by the definitions of C^M and C^M_G, the right square commutes.

6. Quotient Chern classes

6.1. Canonical constructible functions. We introduce naturally defined *canonical constructible functions* and the corresponding equivariant Chern classes for a G-variety.

We again abuse the (set-theoretic) notation for simplicity, and we denote

$$\Psi : G \times X \to X \times X, \ \ \Psi(g, x) = (g.x, x),$$
$$Z := \Psi^{-1}\Delta_X = \{(g, x) | g.x = x\}.$$

The projections of Z to factors are denoted by $G \stackrel{p_1}{\leftarrow} Z \stackrel{q_1}{\rightarrow} X$. Note that $p_1^{-1}(g) = \{x \in X | g.x = x\} =: X^g$ (the fixed point set of the map $g : X \to X$ $(x \mapsto g.x)$) and $q_1^{-1}(x) = \{g \in G | gx = x\} =: Stab_x(G)$ (the stabilizer subgroup of x). Then we can define a G-invariant constructible function on X

$$\alpha^{(1)}_{X/G} := (q_1)_* 1\!\!1_{Z_{red}} \ \in \ \mathcal{F}^G_{inv}(X).$$

The invariance is clear. In particular, for $x \in X$

$$\alpha^{(1)}_{X/G}(x) = \int_{G \times \{x\}} 1\!\!1_{Z_{red}} = \chi(Stab_x(G)_{red}).$$

The adjoint action of G on itself and the preimage of the diagonal are denoted by

$$\Phi : G \times G \to G \times G, \ \ \Phi(h, g) = (hgh^{-1}, g),$$
$$Com(G) = \Phi^{-1}\Delta_G = \{(h, g) | gh = hg\},$$

and the projection $r_2 : Com(G) \to G$ $(r_2(h, g) = g)$. Of course, the fibre $r_2^{-1}(g)$ is isomorphic to the centralizer subgroup $C(g)$ of g. Let

$$Z^{(2)} := \{ \ (h, g, x) \in G \times G \times X \mid gh = hg, \ g.x = x, \ h.x = x \ \}$$

and the natural projections $p_2 : Z^{(2)} \to Com(G)$ $((h, g, x) \mapsto (h, g))$ and $q_2 : Z^{(2)} \to Z$ $((h, g, x) \mapsto (g, x))$. Then for a point $(g, x) \in Z$, the fibre $(q_2)^{-1}(g, x)$ is isomorphic to $Stab_x(G) \cap C(g)$ $(=Stab_x(C(g))$ for the action of $C(g)$ on $X^g)$. It is also obvious that for $(h, g) \in Com(G)$, $(p_2)^{-1}(h, g)$ corresponds to $X^{\{h,g\}} := X^h \cap X^g$, the intersection of fixed point sets. We set $\pi_2 := q_1 \circ q_2 : Z^{(2)} \to X$ and define

$$\alpha^{(2)}_{X/G} := (\pi_2)_* 1\!\!1_{Z^{(2)}_{red}} \in \mathcal{F}^G_{inv}(X).$$

This procedure can be continued by introducing the set of mutually commuting k-tuple of elements of G, $Com(G; k) := \{(g_k, \cdots, g_1) | g_i g_j = g_j g_i\}$ (cf. [5]) and the correspondence

$$Z^{(k)} := \{ \ (g_k, \cdots, g_1, x) \in Com(G; k) \times X \mid g_i \in Stab_x(G) \ \}$$

(e.g., $Com(G;1)=G$, $Com(G;2)=Com(G)$, $Z^{(0)}=X$, $Z^{(1)}=Z$). The natural projections to $Com(G;k)$ and $Z^{(k-1)}$ are defined, thus we have the following commutative diagrams

$$\begin{array}{ccccccc} Z^{(k)} & \stackrel{q_k}{\to} & Z^{(k-1)} & \stackrel{q_{k-1}}{\to}\cdots\stackrel{q_2}{\to} & Z & \stackrel{q_1}{\to} & X \\ p_k\downarrow & & p_{k-1}\downarrow & & p_1\downarrow & & p_0\downarrow \\ Com(G;k) & \stackrel{r_k}{\to} & Com(G;k-1) & \stackrel{r_{k-1}}{\to}\cdots\stackrel{r_3}{\to} & G & \to & \{e\} \end{array}$$

The preimage $p_k^{-1}(g_k,\cdots,g_1)$ is the set of simultaneous fixed points, denoted by $X^{\{g_k,\cdots,g_1\}}$. Let $\pi_0=id_X$ and $\pi_k:=q_1\circ\cdots\circ q_k: Z^{(k)}\to X$. Then the k-th *canonical constructible functions* of X is defined by $\alpha^{(0)}_{X/G}:=1\!\!1_X$ and

$$\alpha^{(k)}_{X/G}:=(\pi_k)_*1\!\!1_{Z^{(k)}_{red}}\quad\in\ \mathcal{F}^G_{inv}(X).$$

Applying our natural transformation to these canonical constructible functions, we obtain a sequence of *integral* Chern classes $C_*(\alpha^{(k)}_{X/G})$ in $H^G_*(X)$, say *canonical quotient Chern classes* for X/G. Furthermore, if $\chi(G)\neq 0$, then *rational canonical constructible functions* are defined by

$$1\!\!1^{(k)}_{X/G,orb}:=\frac{\alpha^{(k)}_{X/G}}{\chi(G)}\quad\in\quad\mathcal{F}^G_{inv}(X)\otimes\mathbb{Q}$$

as well *rational* quotient Chern classes $C_*(1\!\!1^{(k)}_{\mathcal{X},orb})$ in $H^G(X)\otimes\mathbb{Q}$. For the importance of the case $k=1,2$, we write as $1\!\!1_{X/G,quo}:=1\!\!1^{(1)}_{X/G,orb}$ and $1\!\!1_{X/G,orb}:=1\!\!1^{(2)}_{X/G,orb}$.

Note again that the canonical quotient Chern classes are defined for arbitrary actions of G ($\dim G\geq 0$). In the next subsection we will see these classes in a typical case of a finite group action.

6.2. Orbifold Chern class. Let G be a finite group and X a possibly singular quasi-projective variety with G-action. We also assume that X/G is a variety and the stabilizer subgroups are reduced. Combining Proposition 5.6 with the pushforward property of the ordinary C_* for the projection $\pi: X\to X/G$, we have the commutative diagram:

$$\begin{array}{ccc} \mathcal{F}^G_{inv}(X) & \stackrel{C^G_*\circ\phi_0}{\longrightarrow} & H^G_*(X) \\ \pi_*\downarrow & & \downarrow\pi_*\circ i^* \\ \mathcal{F}(X/G) & \stackrel{C_*}{\longrightarrow} & H_*(X/G) \end{array}$$

Since $g=\dim G=0$, the map $\pi_*\circ i^*$, for nonnegative dimension $*\geq 0$, is an isomorphism after tensored by $\mathbb{Q}$, which coincides with the isomorphism mentioned in Remark 3.4 (i.e., Theorem 3 in [6]). Note that the first horizontal arrow is just C_* given for the quotient stack $[X/G]$ in Theorem 3.5, and the above diagram says the simple fact that if X/G is a variety, the above C_* is identified with ordinary MacPherson's transformation (the lower horizontal arrow) within rational coefficients.

An elementary computation shows that

$$1\!\!1_{X/G,quo} = \frac{1}{|G|}\sum_{g} 1\!\!1_{X^g}, \quad 1\!\!1_{X/G,orb} = \frac{1}{|G|}\sum_{gh=hg} 1\!\!1_{X^{\{g,h\}}}$$

in $\mathcal{F}^G_{inv}(X)\otimes\mathbb{Q}$, where the first sum runs over all $g \in G$ and the second runs over all $(h,g) \in Com(G)$: In fact,

$$\begin{aligned} \alpha^{(1)}_{X/G}(x) &= |Stab_x(G)| = \sum 1\!\!1_{X^g}(x); \\ \alpha^{(2)}_{X/G}(x) &= \sharp\,(\pi^{(2)})^{-1}(x) = \sharp\,\{(g,h) | gh = hg, g.x = x, h.x = x\} \\ &= \sum 1\!\!1_{X^{\{g,h\}}}(x). \end{aligned}$$

We call $C^G_*(1\!\!1_{X/G,quo})$ *the Chern class of* X/G and $C^G_*(1\!\!1_{X/G,orb})$ *the orbifold Chern class of* X/G, in abusing words. Note that if X is nonsingular, we can take the dual of these classes in $H^*_G(X)\otimes\mathbb{Q}$.

Proposition 6.1. *It holds that* $\pi_*(1\!\!1_{X/G,quo}) = 1\!\!1_{X/G}$ *and*

$$\pi_* i^* C^G_*(1\!\!1_{X/G,quo}) = C^{SM}_*(X/G) \quad \in \quad H_*(X/G;\mathbb{Q})$$

where i^* *is the map induced by a restriction* i *given in 4.3 and* π_* *is the pushforward induced by the projection* π.

Proof: This is also an elementary computation. Since $|G| = \sharp\, G/Stab_x(G) \cdot |Stab_x(G)| = \sharp\, G.x \cdot |Stab_x(G)|$, and $|Stab_{x'}(G)| = |Stab_x(G)|$ (for $x' \in G.x$), we have that for any $[x] \in X/G$

$$\begin{aligned} \pi_*(1\!\!1_{X/G,quo})([x]) &= \int_{G.x} 1\!\!1_{X/G,quo} = \sum_{x'\in G.x} 1\!\!1_{X/G,quo}(x') \\ &= \frac{1}{|G|}\sum_{x'\in G.x}\sum_{g\in G} 1\!\!1_{X^g}(x') = 1. \end{aligned}$$

The second equality follows from the commutative diagram as described above.

Corollary 6.2. *It holds that*

$$\begin{aligned} \chi(X/G) &= \textit{degree of } 0\textit{-th component } C^G_0(1\!\!1_{X/G,quo}) \\ (&= \int_X 1\!\!1_{X/G,quo} = \frac{1}{|G|}\sum_g \chi(X^g) \quad). \end{aligned}$$

Proof: This follows from the above proposition and the subsection 4.1. Let us take $f : X/G \to \{pt\}$ and $f\circ\pi : X \to \{pt\}$. Then by using the above proposition, we have

$$\int_X 1\!\!1_{X/G,quo} = (f\circ\pi)_*(1\!\!1_{X/G,quo}) = f_*\pi_*(1\!\!1_{X/G,quo}) = f_* 1\!\!1_{X/G} = \chi(X/G).$$

Next, let us look at *the orbifold Euler characteristics* of the quotient X/G, then the same type equalities hold: For a possibly singular G-variety X,

$$\begin{aligned} \chi(X;G) &= \text{degree of 0-th component } \ C_0^G(1\!\!1_{X/G,orb}) \\ (\quad &= \int_X 1\!\!1_{X/G,orb} = \frac{1}{|G|}\sum_{gh=hg} \chi(X^{\{g,h\}}) \quad). \end{aligned}$$

We may expect that $\chi(X;G)$ is related to the Euler characteristics of some certain desingularizations of X/G (cf. [13]). Here is an optimistic conjecture:

Conjecture 6.3. *Let G be a finite group, or more generally a linear algebraic group. Let $X \to Y := X/G$ be the quotient. Then there is a G-variety $\tilde{X}$ and a proper G-morphism $f : \tilde{X} \to X$ so that $f_*^G(1\!\!1_{\tilde{X}/G,quo}) = 1\!\!1_{X/G,orb}$. In particular, $f_*^G C_*^G(1\!\!1_{\tilde{X}/G,quo}) = C_*^G(1\!\!1_{X/G,orb})$ and their 0-th degrees $\chi(\tilde{X}/G) = \chi(X;G)$*

Also we obtain a certain sequence of the generalized orbifold Chern classes $C_*^G(1\!\!1^{(k)}_{X/G,orb})$, whose 0-th component is just the generalized orbifold Euler characteristics defined in [5]:

$$\text{degree of } C_0^G(1\!\!1^{(k)}_{X/G,orb}) = \frac{1}{|G|}\sum_{Com(G;k)} \chi(X^{\{g_1,\cdots,g_k\}}),$$

where $X^{\{g_1,\cdots,g_k\}}$ is the simultaneously fixed point set. As a particularly interesting case, the Chern class of symmetric products ($S^nX = X^n/S_n$, S_n being the n-th symmetry group) is studied in [22], in which *the generating functions of orbifold Chern classes* of S^nX is obtained.

Our principle is that *certain kinds of formulas on Euler characteristics should admit the Chern class version.* From this viewpoint, "the constructible function-description" is seemingly very useful and very natural. Also we have another advantage that there is no trouble even when we deal with singular varieties with G-action.

7. Thom polynomials

In this section we discuss on G-characteristic classes associated to G-classifications of a G-space (=classifications of "singularities" of various objects). This correspondence is given by a "Segre-version" of C_*^G. We assume $k = \mathbb{C}$ in Subsection 7.1, but in the other subsection we think of both contexts although we write H_* throughout.

7.1. Thom polynomial. Let X be a nonsingular G-variety. Then, we have

$$Dual_G \circ C_*^G : \mathcal{F}^G_{inv}(X) \to H^*_G(X)$$

by using G-Poincaré dual. For a G-invariant subvariety W of X with codimension l, the leading term of $Dual_G \circ C_*^G(1\!\!1_W)$ is the G-equivariant Poincaré dual to $[W]_G$ in X, which is usually called *the Thom polynomial* of W (in G-classification of X), denoted by $tp(W) \in H^{2l}_G(X)$, cf. [14], [8] and their references.

In particular, if X is a G-affine space (as a usual case in tp theory) , $H^*_G(X) = H^*(BG)$ and hence $tp(W)$ is written as a polynomial of characteristic classes c_i of G-bundles, which has the "universality" in the following sense:
(Universality): For any bundle $E \to M$ with fibre X and the structure group G over a nonsingular base space M of dimension m, we associate a subbundle $E_W \to M$ with fibre W (because W is G-invariant). For a "generic" section $s : M \to E$, we set

$$W(s) := s^{-1}(E_W)$$

and call this *the singular set of type* W, which has the expected codimension $l = \operatorname{codim} W$. Let $i : W(s) \to M$ be the inclusion. Then, the fundamental class of the singular set is expressed in M by

$$i_*[W(s)] = tp(W)(c(E)) \frown [M] \in H_{2(m-l)}(M)$$

after substituting $c_i(E)$ to c_i arising in $tp(W) \in H^{2l}(BG)$.

This theorem basically goes back to R. Thom [27], and it is proved in a topological setting (cf. [14], [8]). In an algebraic setting, we show a more general statement later.

As a typical example of Thom polynomials, there is the so-called *Thom-Porteous formula*: Let X be the affine space $Hom(\mathbb{C}^m, \mathbb{C}^{m+k})$ on which the group $G = GL(m, \mathbb{C}) \times GL(n+k, \mathbb{C})$ operates from the right and left as linear coordinate changes. The invariant subvariety W under consideration is the closure of the orbit with kernel dimension i, usually denoted by $\bar{\Sigma}^i$. Let $f : E \to F$ be a suitably generic vector bundle map over M, i.e., a section $f : M \to Hom(E, F)$, where E and F are of rank m and $m+k$, respectively. Then the fundamental class of the degenerate loci $\bar{\Sigma}^i(f)$ is expressed (as in cohomology of M) by a certain Schur polynomial in $c_i(F - E)$, that is $tp(\bar{\Sigma}^i)$, cf. [9] Chap. 14. In this section concerning tp, it would be helpful to take this example in mind throughout.

7.2. Generic morphisms with respect to a subvariety. Throughout this subsection, we forget the group action. Let $k = \mathbb{C}$ and f be a morphism of a possibly singular variety into a nonsingular variety, and W a subvariety of the target variety of f. Note that any corresponding complex analytic variety admits a Whitney stratification. We say f is transverse to W (or say, generic with respect to W) if the restriction of f to any stratum of the source variety is transverse to any strata of W.

Instead, we define the "genericity" of f with respect to W in the algebraic context. It is, in fact, related to the VRR theorem. Let us remind that the (ordinary) VRR theorem for *smooth morphisms* f (Yokura [30]) says that C_* is compatible with f^* and $f^{**} = c(\nu_f)^{-1} \frown f^*$, however the theorem fails for non-smooth morphisms (especially, regular embeddings). On one hand, morphisms f in which we are now interested are regular embeddings, or more suitably, local complete intersection morphisms, cf. [9] (a *l.c.i. morphisms* $f : X \to Y$ is a composition of a regular embedding $i : X \to N$ and a smooth morphism (e.g., fibrations) $p : N \to Y$; it has the virtual normal bundle $\nu_f := \nu_X - T_p$).

Typical examples we take in mind are sections of vector bundles (e.g., sections $X \to Y := Hom(E,F)$) or any morphisms between nonsingular varieties.

Although the VRR theorem for l.c.i morphisms fails, the VRR formula for a *distinguished* element $1\!\!1_W \in \mathcal{F}(Y)$ makes sense, that is our definition of the "genericity of f with respect to W":

Definition 7.1. *Let $f : X \to Y$ be a l.c.i. morphism with the virtual normal bundle ν. We say that f is generic with respect to a subvariety W of Y if it holds that $C_* \circ f^*(1\!\!1_W) = f^{**} \circ C_*(1\!\!1_W)$, i.e., $i_* C^{SM}(f^{-1}(W)) = c(\nu)^{-1} \frown f^* i_* C^{SM}(W)$ where i_* are induced maps via inclusions.*

Proposition 7.2. *(1) Any smooth morphism $f : X \to Y$ is generic with respect to any subvariety in Y. (2) Any l.c.i. morphism $f : X \to Y$ is generic with respect to Y.*

Proof: This is straightforward from the definition.

Remark 7.3. *It was J. Schürmann who established the generalized VRR formula for l.c.i. morphisms (Theorem 0.1* [25]*), which describes the defect $C_* \circ f^* - f^{**} \circ C_*$ in terms of the generalized vanishing cycle functor. In the following subsections, we may state theorems by using his vanishing cycle functor, instead of assuming the "genericity". We also remark that in the complex case, the "transversality" (in the sense of the stratification theory) implies the "genericity" in the above sense (cf. Proposition 1.3 of* [23], [21], *Corollary 0.1 of* [25]*).*

7.3. Schwartz-MacPherson Segre class. For a closed subscheme Z in a *nonsingular* variety M, we define

$$s^{SM}(Z, M) := c(TM|_Z)^{-1} \frown C^{SM}(Z) \ \in \ H_*(Z)$$

as an analogy to the relation of the Segre covariance class and Fulton's canonical class defined in [9], cf. Remark 5.5. This "Segre version of SM classes" has been introduced also in [2]. We may denote this class by $s^{SM}(1\!\!1_Z, M)$ (Z being a (reduced) subvariety). We also define $s^M(Z, M)$ by replacing C^{SM} to C^M (for a subvariety Z, $s^M(1\!\!1_Z, M) = s^M(Eu_Z, M)$).

Let us return to our equivariant setting. We give a generalization of tp as follows:

Definition 7.4. *For an invariant subvariety W in a nonsingular G-variety X (its G-inclusion is denoted by $i : W \to X$), we define the universal Segre-SM class*

$$s_G^{SM}(W, X) := c^G(TX|_W)^{-1} \frown C_*^G(1\!\!1_W) \ \in \ H_*^G(W),$$

or equivalently, $s_G^{SM}(W, X) = \varphi_U s^{SM}(W \times_G U, X \times_G U)$ where φ_U is the limit map $H_{trunc}(W \times_G U) \to H_*^G(W)$. Its G-equivariant dual in X is denoted by

$$tp^{SM}(W) := Dual_G \, i_*^G s_G^{SM}(W, X) \ \in \ H_G^*(X).$$

Note that $tp^{SM}(W)$ is a formal power series

$$tp^{SM}(W) = \sum_{i=0}^{\infty} tp_i^{SM}(W) \in H_G^*(X) = \prod H_G^i(X).$$

7.4. Universality for sections. Let s be a section of a bundle $E \to M$ with fibre X and structure group G, W an invariant subvariety of X. For short, we say $s : M \to E$ is *generic* (with respect to W) if the morphism s is generic with respect to the associated subbundle E_W with fibre W. In the following theorem, we assume that M is a *quasi-projective* variety.

Theorem 7.5. *Let X be a G-affine space, W an invariant subvariety of X of codimension l. Then,*
(1) $tp_i^{SM}(W) = 0$ $(i < l)$ and $tp_l^{SM}(W)$ coincides with the Thom polynomial $tp(W)$:

$$tp^{SM}(W) = tp(W) + \textit{higher terms}.$$

(2) (universality) For any generic bundle $E \to M$ and any generic section s w.r.t. W, we have

$$i_* C^{SM}(W(s)) = tp^{SM}(W)(c(E)) \frown C^{SM}(M) \ \in \ H_*(M),$$

where i_* denotes the induced map by the inclusion.

Remark 7.6. *As a generalization of Thom-Porteous formula, Parusiński-Pragacz* [23] *give a formula of Schwartz-MacPherson classes of degeneracy loci of bundle maps, that is one of our motivation to define our generalized tp as given above. In fact, they computed tp^{SM} of $\bar{\Sigma}^i$, see Theorem 2.1 in* [23].

Proof: (1) This is obvious, in fact the top term of $i_*^G C_G^{SM}(W)$ is just $i_*^G[W]_G \in H_{2(n-l)}^G(X)$.
(2) The main point is the following key lemma (Lemma 1.6 of Totaro [28]) on the existence of classifying maps of G-bundles over a quasi-projective variety:

Lemma 7.7. *(*[28]*): For any algebraic bundle $E \to M$ with fibre X and structure group G over a quasi-projective variety M, there is a bundle $q : M_1 \to M$ with fibre being an affine space which admits an algebraic classifying map $\rho : M_1 \to U/G$ for some large $U = V - S$ $(\in I(G))$ so that $\rho^*(X \times_G U) \simeq q^*E$.*

$$\begin{array}{ccccccc}
E & \longleftarrow & q^*E & \xrightarrow{\bar{\rho}} & X \times_G U & \to & X \times_G EG \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
M & \xleftarrow{q} & M_1 & \xrightarrow{\rho} & U/G & \to & BG
\end{array}$$

The rest of the proof is straightforward from this lemma, the inductive limit argument and the genericity of morphisms w.r.t. certain varieties associated to W.

We take M_1 as in Lemma 7.7 and denote $M_2 := X \times_G U$ and $W_G := W \times_G U$ ($\subset M_2$). Now X is assumed to be an affine space, so M_2 is nonsingular, and hence $C^{SM}(M_2) = c(TM_2) \frown [M_2]$. Then

$$\begin{aligned}
& r_U\, tp^{SM}(W) \frown C^{SM}(M_2) \\
& \quad = r_U\, tp^{SM}(W) \cdot c(TM_2) \frown [M_2] \\
& \quad = c(TM_2) \cdot r_U \circ Dual_G \circ i_*^G \left(s_G^{SM}(W, X) \right) \frown [M_2] \\
& \quad = c(TM_2) \cdot Dual^{-1} \circ r_U \circ Dual_G \circ \varphi_U(i_* s^{SM}(W_G, M_2)) \\
& \quad = c(TM_2) \frown i_* s^{SM}(W_G, M_2) \\
& \quad = c(TM_2) \frown \left(c(TM_2)^{-1} \frown i_* C^{SM}(W_G) \right) \\
& \quad = i_*\, C^{SM}(W_G).
\end{aligned}$$

The section $s : M \to E$ induces a section $s' : M_1 \to q^*E$, and then set $f := \bar{\rho} \circ s' : M_1 \to M_2$ and ν_f the virtual normal bundle of f (in fact $\bar{\rho}$ can be taken as a regular embedding). Since we assume s is generic (w.r.t. E_W), it turns out that f is generic w.r.t. W_G and hence we have

$$i_*\, C^{SM}(W(s')) = c(\nu_f)^{-1} \frown f^*\, i_*\, C^{SM}(W_G),$$

$$C^{SM}(M_1) = c(\nu_f)^{-1} \frown f^*\, C^{SM}(M_2).$$

Also for the smooth morphism $q : M_1 \to M$,

$$i_*\, C^{SM}(W(s')) = q^{**}\, i_*\, C^{SM}(W(s)), \quad C^{SM}(M_1) = q^{**}\, C^{SM}(M),$$

where $q^{**} = c(T_q) \frown q^* : H_*(M) \to H_*(M_1)$. Since $H_G^*(X) = H_G^*(pt) \simeq H^*(BG)$ (X being a G-affine space), $f^* r_U$ is identified with the pullback via the classifying map $\rho^* r_U$, which sends the universal characteristic class $c_i \in H^{2i}(BG)$ to the Chern class $c_i(q^*E) \in H^{2i}(M_1)$. Thus, we have

$$\begin{aligned}
q^{**}\, i_*\, C^{SM}(W(s)) &= i_*\, C^{SM}(W(s')) \\
&= c(\nu_f)^{-1} \frown f^*\, i_*\, C^{SM}(W_G) \\
&= c(\nu_f)^{-1} \frown f^* \left(r_U\, tp^{SM}(W) \frown C^{SM}(M_2) \right) \\
&= f^*\, r_U\, tp^{SM}(W) \frown \left(c(\nu_f)^{-1} \frown f^*\, C^{SM}(M_2) \right) \\
&= tp^{SM}(W)(c(q^*E)) \frown C^{SM}(M_1) \\
&= tp^{SM}(W)(c(q^*E)) \frown q^{**}\, C^{SM}(M) \\
&= q^{**} \left(tp^{SM}(W)(c(E)) \frown C^{SM}(M) \right).
\end{aligned}$$

Since q^{**} is an isomorphism, this equality shows (2).

Corollary 7.8. *Let $k = \mathbb{C}$. For a generic section s as above, the topological Euler characteristic of the singular set $W(s)$ of type W is given by*

$$\chi(W(s)) = \int_M tp^{SM}(W)(c(E)) \frown C^{SM}(M).$$

Proof: This is a direct consequence from the fact that the Euler characteristic of a variety is equal to the 0-th degree of its (ordinary) Schwartz-MacPherson class.

Remark 7.9. *We summarize this section. As seen in §6, our equivariant MacPherson theory C_*^G certainly gives the Chern class version of various Euler characteristics of quotients, while in this section, viewing from the top dimensional side, our C_*^G (precisely its Segre-version) gives "higher dimensional" Thom polynomial theory.*

For a nonsingular G-variety X, we have introduced universal Segre-SM classes of invariant subvarieties, that produces a homomorphism between abelian groups

$$tp^{SM} : \mathcal{F}^G_{inv}(X) \to H^*_G(X).$$

This is actually a natural transformation for the category of nonsingular G-varieties and proper G-morphisms. In particular, if X be a G-affine space, the values are universal polynomials in Chern classes for G-bundles. In other words, tp^{SM} is a correspondence from a "local G-classification" of singularities to "global invariants" for generic sections of any associated affine bundles $E \to M$, that we called the "universality" of tp^{SM} throughout this section. The property is shown totally in the algebraic context, by using Totaro's classifying maps. Besides, we may define tp^M and tp^F in the same way as tp^{SM}, the relations among which should be related to local invariants of closures of G-orbits (e.g., local Euler obstruction).

As a typical example, let us explain this "local-global" correspondence in Singularity theory of differentiable mappings between manifolds: The classification problem of map-germs is reduced to the classification in the level of some r-jets, i.e., the so-called finite determinacy. So then we think of some r-jet space $X = J^r(m, m+k)$ of germs $\mathbb{C}^m, 0 \to \mathbb{C}^{m+k}, 0$ (the space of r-th Taylor expansions) with the action of the group G of r-jets of coordinate changes of source and target (called the $\mathcal{RL}$ (right-left) classification). An invariant subvariety (e.g., the closure of an orbit or of a family of orbits) is called a singularity type. The case of $r = 1$ is just singularities $\bar{\Sigma}^i$ of vector bundle maps. For any suitable generic map $f : M^m \to N^{m+k}$, a singularity type η in the local classification gives a "global invariant of f" $tp(\eta)$ and $tp^{SM}(\eta)$, which universally express the fundamental class and the Chern class of $\eta(f)$ (Precisely saying, the $\mathcal{K}$-classification is much useful for the classification of generic singularities of maps, then a further inductive limit process arises according to the parameters $r \to \infty$ and $m \to \infty$ with fixed difference k of the dimension of source and target spaces. It turns out that tp (each component of tp^{SM}) is a polynomial in c_i's which mean Chern classes of the virtual normal bundle $c_i(f^*TN - TM)$ for any generic $f : M \to N$, see [14], [8], [21]*). There may be many interesting directions for further researches,*

for instance, computational aspects of tp^{SM}*,* tp^{M} *and* tp^{F} *in relation with local invariants such as Milnor number and Polar multiplicities, and the* tp^{SM}*-version for multi-singularities (* [15]*) combined with materials in* §*6, etc.*

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